LEMMATA IN ADIC-RIGID GEOMETRY

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1. INTRODUCTION

This is a file recording some interesting lemmas/facts/references on adic-rigid geometry the author has come across. Hopefully it could be helpful to someone else on this planet. Some of the Lemmas can easily be extended to analytic adic spaces but the author does not intend to do so[1].

The readers will be assumed to be as (un)-familiar on this subject as the author does. The references will only be given literally but not bibliographically.

2. NOTATIONS AND CONVENTIONS

Throughout the whole file, unless stated otherwise, straight letters denote adic/rigid objects and curly letters denote formal objects. Let K be a non-archimedean field, in each section of the file we will put more conditions (discretely valued or not, characteristics, etc.) on K. We use $\mathcal{O}_K$ to denote the ring of integers in K, $\mathfrak{m}$ to denote the maximal ideal in $\mathcal{O}_K$ and κ the residue field of K.

Let A be an affinoid algebra over K. If no confusion arise, we will denote $\mathrm{Sp}(A)$ by X and $\mathrm{Spa}(A) := \mathrm{Spa}(A, A^\circ)$ by X^{ad}. For an admissible open $U \subset X$, sometimes we still use U to denote U^{ad}. Hopefully this will not confuse the reader.

We will use V to denote a valuation ring and denote its value group by Γ. We will think of the value of a function on an adic point as *norm* (so the value of elements in $\mathfrak{m}$ will be less than 1).

3. VALUATIONS

This section is supposed to record every interesting (random?) statements about valuations.

The last paragraph on page 198 of the book by Fresnel–van der Put is really interesting. Let V be a general valuation ring. Let Γ be the value group of V.

Definition 3.1.
  (1) The *rank* of V is defined as the dimension of $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space and is denoted by $\mathrm{rk}(V)$.
  (2) The *real rank* of V is defined as the dimension of $\widehat{\Gamma} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ as a $\mathbb{R}$-vector space where the completion is taken w.r.t. the obvious induced total ordering ($\widehat{\Gamma}$ is a totally ordered vector space over $\mathbb{R}$).

The following Lemma is stated in the aforesaid paragraph and is not so hard to prove.

Lemma 3.2. *The real rank of V is equal to the Krull dimension of V.*

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[1]After all this is not going to be an adic project, although it is certainly inspired by the Stacks Project.

Remark 3.3. Note that the prime ideals of V are totally ordered by inclusion.

Abhyankar's inequality is useful.

Theorem 3.4 (Abhyankar's inequality)**.** *Let $K \subset L$ be an extension of valued fields, with residue fields $\kappa \subset \ell$ and value groups $\Gamma_K \subset \Gamma_L$. Then we have*

$$tr_\kappa(\ell) + \dim_\mathbb{Q}(\mathbb{Q} \otimes_\mathbb{Z} (\Gamma_L/\Gamma_K)) \le tr_K(L).$$

By definition we can verify that the real rank is bounded above by rank. Therefore we have the following Corollary.

Corollary 3.5. *Let V be a valuation ring with $\mathrm{Frac}(V) = F$ being a valued extension of a valued field K. Then we have*

$$\dim(V) = real\ rank(V) \le \mathrm{rk}(V) \le tr_K(F) + \mathrm{rk}(K).$$

The estimation above is very crude in the sense that the transcendence degree of F over K is often gigantic. One may actually replace it by a dense subfield.

Lemma 3.6. *In the situation above, if we can find $\{v_\lambda \in V\}_{\lambda \in \Lambda}$ such that $K(v_\lambda) \subset F$ is dense w.r.t. the topology coming from V. Then we have*

$$\mathrm{rk}(V) \le |\Lambda| + \mathrm{rk}(K).$$

Let us get back to the setup of classical rigid geometry. Using Noether's normalization for affinoid algebras and the fact that $K[X_i]$ is dense in $K\langle X_i\rangle$, one can prove the following Corollary.

Corollary 3.7. *Let X be an irreducible n-dimensional affinoid space. Then the adic points have rank at most $n + 1$.*

Remark 3.8. We will (somewhat) describe all the rank $n + 1$ points in Section 5.

4. Book by Fresnel and van der Put

4.1. **7.1.** In this subsection, we focus on questions/exercises/facts regarding the subsection 7.1 in the book (English version) by Fresnel and van der Put.

Now we start fighting the Exercises 7.1.12 (wide open neighborhoods). Let us recall that to an affinoid space $X := \mathrm{Sp}(A)$, one can associate its adic spectrum $\mathcal{P}(X) = X^{ad}$ and Berkovich's spectrum $\mathcal{M}(X)$. There is a quotient map (as topological spaces) which we denoted as

$$r : X^{ad} \to \mathcal{M}(X).$$

Lemma 4.1 (7.1.12. (1))**.** *For any rank 1 point $a \in \mathcal{M}(X)$, the closure of $\{a\}$ in X^{ad} is the set $r^{-1}(\{a\})$.*

Proof. Since $\mathcal{M}(X)$ is a Hausdorff and r is continuous, we see that $r^{-1}(\{a\})$ is a closed set containing a. Therefore it suffices to show that for any point $p \in r^{-1}(\{a\})$ and any rational $X\langle\frac{f_i}{f_0}\rangle$ containing p, it contains a also. This just follows from the following Lemma which is obvious. $\square$

Lemma 4.2. *Let p, q be two valuations on A and f, g be two elements in A. Suppose p generalizes to q. If $|f|_p \le (\ge)|g|_p$ then $|f|_q \le (\ge)|g|_q$.*

Proof. Obvious? $\square$

Definition 4.3 (Wide open neighborhood of a point). Let $a \in \mathcal{M}(X)$. An admissible W will be called a *wide open neighborhood* of a if it contains a rational $X\langle\frac{f_i}{1}\rangle$ with $|f_i|_a < 1$.

Lemma 4.4 (7.1.12. (2)). *The W above is a wide open neighborhood of a if and only if it contains every point $p \in X^{ad}$ such that $r(p) = a$ (in other words, contains the closure of $\{a\}$ by Lemma 4.1).*

Proof. The only if part follows easily from Lemma 4.2.

Now we try to prove the if part. For every $f \in A$ such that $|f|_a < 1$ let us consider the rational $U_f := X\langle\frac{1}{f}\rangle$ in X. We claim that the complement (in X^{ad}) to the union of U_f^{ad} is just the closure of $\{a\}$. Assuming this claim we see that U_f^{ad}'s along with W^{ad} form an open cover of X^{ad} which is quasi-compact by Huber. Therefore there are finitely many functions $f_i \in A$ with $|f_i| < 1$ such that $\{W, U_{f_i}\}$ is an admissible cover of X. After replacing f_i by some big power, we may assume that $|f_i|_a < |\pi|$ for some $\pi \in \mathfrak{m}$. Let $U = X\langle\frac{f_i}{\pi}\rangle$, we see that $U \cap U_{f_i} = \varnothing$. Therefore $U \subset W$. So we are done if we replace f_i by $\frac{f_i}{\pi}$.

As for the claim, suppose $p \in X^{ad}$ is in the complement. Consider $b := r(p)$. By the assumption, we see that for any element $f \in A$ such that $|f|_a < 1$ we have $|f|_b \leq 1$. By the next Lemma 4.5, we see that $a = b$. $\square$

Lemma 4.5. *Let a, b be two rank 1 valuations on A such that for any element $f \in A$ such that $|f|_a < 1$ we have $|f|_b \leq 1$. Then $a = b$.*

Proof. It suffices to show that $|f|_a < 1$ actually implies $|f|_b < 1$. We see that $|f^N|_a < |\pi|$ for big enough N. And the condition implies that $|\frac{f^N}{\pi}|_b \leq 1$, which implies that $|f|_b < 1$. $\square$

Lemma 4.6 (7.1.12. (3)). *Suppose that $\{X_i\}_{i \in I}$ are affinoid subsets of X such that for every rank 1 point $a \in X^{ad}$ there exists an i with X_i^{ad} a wide open neighborhood of a. Then there exists a finite subset $J \subset I$ such that $X = \bigcup_{j \in J} X_j$ (namely, they form an admissible cover).*

Proof. By Lemma 4.4, we see that $\{X_i^{ad}\}$ is a cover of X^{ad}. Therefore, due to quasi-compactness of X^{ad}, finitely many of X_i^{ad}'s would cover X^{ad}. $\square$

Definition 4.7 (Wide open neighborhood of an affinoid). Let $U \subset V$ be two quasi-compact opens in X^{ad} (namely, they are adic analytification of two admissible which are finite union of rational subsets of X). We call V a *wide open neighborhood* of U (in X) if V is a wide open neighborhood of a for any $a \in \mathcal{M}(U)$.

Lemma 4.8 (7.1.12. (4)). *Let notations be as above. The following are equivalent:*

(1) *V is a wide open neighborhood of U;*
(2) *$U \subset\subset_X V$;*
(3) *$\overline{U} \subset V$;*
(4) *$\mathcal{M}(V)$ is a neighborhood of $\mathcal{M}(U)$.*

Remark 4.9. See also Schneider's paper "Points of rigid analytic varieties".

Proof. (1), (2) and (4) are equivalent due to Proposition 23 of Schneider's paper mentioned above. The equivalence of (1) and (3) follows from Lemma 4.10 below. $\square$

Lemma 4.10. *Let $U \subset X$ be an admissible open. Then $\overline{U^{ad}} \supset r^{-1}(\mathcal{M}(U))$. Assume further that $U \subset X$ is a finite union of rational subsets. Then we have $\overline{U^{ad}} = r^{-1}(\mathcal{M}(U))$.*

Proof. If $p \in X^{ad}$ satisfies $r(p) \in \mathcal{M}(U)$ and let V be an arbitrary open of p. Then trivially $r(p) \in U \cap V$. Hence $p \in \overline{U^{ad}}$.

The further condition implies that $\mathcal{M}(U)$ is closed in $\mathcal{M}(X)$. Therefore $\overline{U^{ad}} \subset r^{-1}(\mathcal{M}(U))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is easy to see that the cofinal system of wide open neighborhoods satisfy the following properties.

Lemma 4.11 (7.1.12. (5))**.**

(1) *Let $Y = X\langle \frac{g_i}{g_0} \rangle \subset X$ be a rational subset. Then one defines the rational subset $Y(r) := X\langle \frac{g_i}{rg_0} \rangle^2$ for any $r \in \sqrt{|k^*|}, \quad r > 1$ (OK, the notation is wrong somehow since r is not an element in A. Hopefully my dear reader would understand what I meant). Then the $\{Y(r)\}$ forms a cofinal family of wide open neighborhoods of Y. In particular, any two different presentations of Y give rise to cofinal family of admissible opens.*
(2) *If Y is union of finitely many rational subsets Y_i, then the family of unions of $Y_i(r)$'s form a cofinal family of wide open neighborhoods of Y.*
(3) *Let Y_i be finitely many rational subsets of X. Then $\bigcap Y_i(r)$ forms a cofinal system of wide open neighborhoods of $\bigcap Y_i$.*

Let $U \subset V$ be two arbitrary admissible opens of X. The exercise 7.1.12. (6) asks us to generalize the notion of wide open neighborhood for an arbitrary admissible open. The author overthought about this question and was wondering whether he should define it as $\overline{U^{ad}} \subset V$ or $r^{-1}(r(U)) \subset V$. Later we will see it is more reasonable to use the latter definition (which is just saying for every rank 1 point in U, V is a wide open neighborhood of it). Let us first notice the following

Lemma 4.12. *Let notations be as above. Then we have*

$$V \supset \overline{U} \Rightarrow V \supset r^{-1}(r(U)).$$

Proof. This just follows from Lemma 4.16. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The other implication does not hold in general. To illustrate this, let us notice the following

Example 4.13. Let $U = V$ be the punctured (at "origin") disc inside closed disc of radius 1. We see that $\overline{U}$ is the whole closed disc whereas $r^{-1}(r(U)) = U$.

The discussion above justifies (at least it makes sense to the author) the following

Definition 4.14. Let notations be as above. We say V is a wide open neighborhood of U if for every rank 1 point $a \in U$, we have $r^{-1}(\{a\}) \subset V$.

Two exercises left, maybe the author will get back to them later.

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[2]This is called an inflated open of Y in the author's translation of Bartenwerfer's paper.

4.2. **7.2.** In this section, we focus on questions/exercises/facts regarding the subsection 7.2 in the book (English version) by Fresnel and van der Put.

Let us verify the second-to-last sentence in the proof of Lemma 7.2.1 in the aforesaid book. By definition

$$F_a(x) = |z - x|_{|\cdot|_F} = \inf \left\{ \|z - x\|_{D(r)} \,\Big|\, r \in [0,1] \text{ and } D(r) \neq \varnothing \right\}.$$

If $r \geq F(x)$ then $x \in D(r)$ and $\|z - x\|_{D(r)} = r$. If $r < F(x)$ then $x \notin D(r)$ and for every $y \in D(r)$ we have $F(y) < F(x)$, hence $|y - x| = F(x)$ for every $y \in D(r)$. The analysis above tells us $F_a(x) = F(x)$.

Example 4.15 (7.2.5)**.** Let K be an algebraically closed non-archimedean field of characteristic $p > 0$. Let $X = \mathrm{Sp}(K\langle T \rangle)$. Consider $h : X \to X$ given by $h(T) = T^p - T$. The induced map $\mathcal{M}(X) \to \mathcal{M}(X)$ ramifies (topologically but not arithmetically) only at the Gauss point. It is easy to see that any point other than Gauss point is not mapped to Gauss point.

First we observe that for any rank 1 point a other than the Gauss point, the set $h^{-1}(a)$ has exactly p points. One way to see this is by noticing

$$\{(x + \epsilon)^p - (x + \epsilon) \quad | \quad |\epsilon| \leq r\} = \{x^p - x + \epsilon \quad | \quad |\epsilon| \leq r\}$$

where $r \leq 1$ is any real number. We also need to use the fact that h restricted to the classical points are unramified along with Krasner's Lemma.

It's evident that the Gauss point is mapped to itself under h. We just observe that h induces the following map on residue field (of the residue field of the Gauss point)

$$\overline{h} : \kappa(t) \to \kappa(t) \quad \overline{h}(t) = t^p - t,$$

where t is the residue of T. This map of residue field is of degree p. So we see that the induced map is topologically ramified only at the Gauss point. But since h preserves the supremum norm and $\overline{h}$ is separable, we see that h is arithmetically unramified at the Gauss point.

Actually the discussion above extends to the adic space associated with X. On the rank 2 points, one just gets Artin-Schreier map of $\mathbb{P}^1$ or $\mathbb{A}^1$ depending on the generization of the rank 2 point. So the induced map on X^{ad} only ramifies (topologically but not arithmetically) at the Gauss point.

For the rest of this subsection, we want to focus on the behavior of taking closure (inside adic space) of an arbitrary admissible open. Let $U \subset X$ be an admissible open. Then we have

Lemma 4.16.
$$r^{-1}(r(U)) \subset \overline{U} \subset r^{-1}(\overline{r(U)}).$$

Proof. For any rational subset U' of X contained in U, we have $r^{-1}(r(U')) = \overline{U'} \subset \overline{U}$ due to Lemma 4.10. The first inclusion now follows from the fact that $r(U)$ is just the union of $r(U')$'s.

Second inclusion is even easier as r is a continuous map. $\square$

In general both inclusions are not equality. We illustrate this by the following

Example 4.17. Let X be the closed disc of radius 1. Let $U = \{|x| < 1\}$. Then $r^{-1}(r(U)) = U$ since U is evidently a "tube". Whereas $\overline{U} = U \cup \{p\}$, here the valuation $p(T)$ is smaller than 1 but greater than any real number which is < 1. Lastly $r^{-1}(\overline{r(U)}) = U \cup r^{-1}(\{\text{Gauss point}\})$.

4.3. **7.3.** In this section, we focus on questions/exercises/facts regarding the sub-section 7.3 in the book (English version) by Fresnel and van der Put.

The following is proved in "Etale cohomology of rigid analytic spaces" by de Jong and van der Put.

Theorem 4.18. *Let X be a quasi-separated paracompact rigid space. Let F be an abelian sheaf on X. Then the Čech cohomology and sheaf cohomology of F agree.*

In general we can only get the zeroth and first cohomology being the same. The author has not read the paper in loc.cit. yet, hopefully he will find some time to do so.

4.4. **7.4.** In this section, we focus on questions/exercises/facts regarding the sub-section 7.4 in the book (English version) by Fresnel and van der Put.

The following is Theorem 2.7.4 in "Etale cohomology of rigid analytic spaces" by de Jong and van der Put. See also Theorem 7.4.5 of the book that the author is following.

Theorem 4.19. *Let $f\colon Y \to X$ be a quasi-compact morphism of rigid analytic varieties over k. Let a be a rank 1 point on X^{ad} and let F be a sheaf on Y. Then we have*

$$(R^i f_*(F))_a \cong H^i(Y_a, F).$$

In the case when F is overconvergent, there is an easy proof. In this case, we may argue using their associated Berkovich's space and then the associated map f^{Ber} is proper (in the topological sense). Therefore this theorem follows from the topological base change theorem.

This Theorem is useful when one considers the sheaf of topologically nilpotent functions.

5. Points in terms of algebraic geometry

As a student of an algebraic geometer, the author would like to make an effort on understanding adic points in an algebro-geometric way. This section records some of his naïve thoughts originated from several conversations with his advisor.

Recall the following well-known theorem (see, for instance, Proposition 7.3.3 of the book by Fresnel–van der Put).

Theorem 5.1. *Let X be an affinoid space. Then the associated adic space is homeomorphic (as a topological space) to the inverse system of formal models of X.*

Lemma 5.2 (see also Wedhorn's notes on adic spaces, Remark 4.12 and Remark 7.40). *For every adic point of X, its generization (as a topological space) will be a totally ordered chain of finite length. Also, every adic point of X has a unique rank 1 generization.*

5.1. **Closed disc.** In this subsection, we look closely at the example of closed disc of radius 1 over an algebraically closed K.

There are five types of points of X^{ad}. See also Example 2.20 of Scholze's paper "Perfectoid Spaces". In the language of inverse limit of formal models, one can actually distinguish these five types in the following way. In fact we can distinguish them when we just considering formal models as formal admissible blow ups of $\mathrm{Spf}(\mathcal{O}_K\langle T\rangle)$ at ideals of the form $(T - a, \pi)$ where $a \in \mathcal{O}_K$ and $\pi \in \mathfrak{m}$.

The Gauss points (called type (2) in loc. cit.) are just generic points of components of formal models being considered above. The rank 2 points (called type (5) in loc. cit.) are points on the associated Riemann-Zariski spaces. They can be characterized as following: their image in every formal model are singular and their image stays on a fixed component (in a cofinal system of formal models). These will be called Nostalgic points in this notes.

The points of type (3) can be characterized as following: their image in every formal model are singular and their image do not stay on a fixed component.

The points of type (1) or (4) has image sometimes smooth and closed on formal models. The difference of them being that for a point P of type (4), there exists an $1 \geq r \in \Gamma_K$ such that whenever we consider its image $P_{a,\pi}$ in a further blow up given by $(T - a, \pi)$ with a specialized to the same image as P and $|\pi| < r$ then $P_{a,\pi}$ is always a singular point[3]. Whereas the classical points has no r as above.

One can compare also with what Zariski already knew (in 1939!).

5.2. **Nostalgia.** By flatness + locally topologically finitely presentedness (ltfp), the central fiber of a formal model of an irreducible n-dimensional affinoid space will be equi-n-dimensional. Combining this with Theorem 3.4, we can describe all of the rank $n + 1$ points we promised in Remark 3.8.

Let A be an n-dimensional irreducible affinoid algebra with X its associated affinoid space. Let η be a generic point of an irreducible component in the central fiber of a formal model of X. Then for any blowup, η (as a regular codimension 1 point on the spectrum of A°) lifts uniquely. Therefore we see that η defines a rank 1 point $|\cdot|_\eta$ (from now on, we will still call it η) of X^{ad}. In terms of valuations, η can be realized as the supremum norm of generic fibre of some irreducible Zariski open U_η containing η. Note that it is not only a semi-norm but rather a norm is exactly due to U_η being irreducible. Let $\kappa(\eta)$ be the residue field of η which is an extension of κ with transcendence degree n. Evidently the Riemann-Zariski space RZ_η[4] associated with $\kappa(\eta)$ maps to X^{ad}. With some effort, one can see that this is exactly $r^{-1}(\{\eta\})$ (in other words, the closure of η). The image of these points in a formal model where η is a generic point is just the irreducible components containing η. This picture/phenomenon/fact motivates the following definition.

Definition 5.3 (non-canonical terminology introduced)**.** A *Gauss point* q of X^{ad} is defined to be a point of the form η as above. A point is called *nostalgic*[5] if its unique rank 1 generization is a Gauss point[6].

Finally, we focus back to the rank $n + 1$ points. Suppose $\text{Spa}(F, V) \to X^{ad}$ is a rank $n + 1$ point. Then we see that there is a formal model $\mathcal{X}$ such that $\text{Spec}(V/\mathfrak{m}) \to \mathcal{X}_0$ is a rank n valuation. Abhyankar's inequality, c.f.Theorem 3.4, now asserts that the image of $\text{Spec}(\text{Frac}(V/\mathfrak{m}))$ is a generic point and hence the image of $\text{Spa}(F, V_\mathfrak{m})$ (the rank 1 generization of $\text{Spa}(F, V)$) is an Gauss point. It is easy to see that every Gauss point has a rank $n + 1$ specialization. Now we come to the following

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[3]The author understands that this expression is vague and obscure but he does not know a better way to say it.

[4]Not to be confused with Rapoport-Zink space which has nothing to do with our story here.

[5]Originally the author was using patriotic instead of nostalgic, but one of his friends convinced him that that is a bad idea.

[6]This definition would make Gauss nostalgic which perhaps is not the case.

Scholium 1. Every rank $n + 1$ point on X^{ad} is nostalgic. A point on X^{ad} is nostalgic if and only it has the same generization as a rank $n + 1$ point.

It is easy to prove the following

Lemma 5.4. *The set of nostalgic points are dense in X^{ad}, but it has no intersection with the set of classical points (which is also dense) unless X is 0-dimensional.*

Now we can describe the behavior of taking closure for quasi-compact open subsets inside (the adic space associated with) an affinoid space.

Lemma 5.5. *Let $U \subset X$ be a finite union of rational subsets. Then there is finitely many Guass points $\eta_i \in U^{ad}$ such that*

$$\overline{U^{ad}} = U \cup \bigcup \overline{\{\eta_i\}}.$$

Proof. Any U above must come from a Zariski open $\mathcal{U}$ of some admissible formal model. Choose η_i to be generic points of $\mathcal{U}$, then we are done by the following Theorem (which can be found on a letter from Deligne to Raynaud). $\square$

Remark 5.6. By the proof we see that the Lemma above holds true for any quasi-compact open inside a rigid space which is the generic fibre of an admissible formal model. This would actually help us to construct a rigid space which is *not* of that form.

Theorem 5.7. *Let $\mathcal{U}, \mathcal{V}$ be two disjoint Zariski open in an admissible formal model $\mathcal{X}$. Then there is an admissible blow up of $\mathcal{X}$ such that the strictly transformation of $\mathcal{U}, \mathcal{V}$ has disjoint closure.*

If we specialize ourselves to the case of 1-dimensional rigid spaces, we get the following.

Lemma 5.8. *Let X be a paracompact quasi-separated rigid curve. Let U be a quasi-compact admissible open. Then $\overline{U^{ad}} \setminus U$ is a finite set.*

Recall that Riemann-Zariski space associated to a transcendental field extension can be realized as inverse limit of "models".

Definition 5.9. The "*structure sheaf*" on Riemann-Zariski space is defined as the colimit of pull back of structure sheaf on the "models".

Let $\eta \in X^{ad}$ be a Gauss point. We know that $\overline{\{\eta\}}$ admits a closed immersion ι into X^{ad}. The structure sheaf on $\mathrm{RZ}_\eta \cong \overline{\{\eta\}}$ now has a natural interpretation via adic-rigid geometry.

Lemma 5.10. *We have a natural isomorphism*

$$\mathcal{O}_{\mathrm{RZ}_\eta} \cong \iota^*(\mathcal{O}^\circ / \mathcal{O}^{\circ\circ}).$$

Here the sheaf $\mathcal{O}^\circ$ (resp. $\mathcal{O}^{\circ\circ}$) is the sheaf of functions with norm ≤ 1 (resp. < 1) on X (remember the topoi of sheaves on X is canonically identified with that on X^{ad}).

6. Pathological Examples

In this section, we record pathological rigid spaces the author has randomly come across.

Example 6.1. Let $f: X \to Y$ be a morphism of rigid spaces. Suppose X is quasi-compact, Y is quasi-separated and f is surjective on classical points. Then Y is also quasi-compact. The assumption on Y being quasi-separated is actually necessary, here is an example.

Let $K = \mathbb{C}_p$ and let X be the closed disc of radius 1. Choose an irrational number r with $0 < r < 1$. Let Y be the gluing of infinitely many closed discs of radius 1, where we glue all the radius greater than r parts together and do the same to all the radius less than r parts. There is an obvious map from X to Y that is a surjection on classical points. It is easy to see that Y is not quasi-compact.

Example 6.2. In this example we give an example of quasi-separated rigid curve which does not admit a formal model. Such an example would necessarily be non-paracompact.

Let $K = \mathbb{C}_p$, and choose an increasing sequence of positive rational numbers $\{r_n\}_n$ tending to 1. Let X be the closed disc of radius 1. For each n we glue the circle of radius 1 of a closed disc D_n of radius 1 to the circle of radius r_n on X. The resulting space Y is quasi-separated (but not paracompact). Let $U \subset Y$ be the annulus of outer-radius 1 and inner-radius $1 - \epsilon$. We see that one need infinitely many Gauss points η_n on Y to make $\overline{U^{ad}}$ to be contained in the union of U and $\overline{\{\eta_i\}}$. Therefore by Remark 5.6 (see also, Lemma 5.6) Y does not admit a formal model.